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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Monday 26-10-2009, 13.00-16.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 16 parts. The 16 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

A spinor field transforms under Lorentz transformations as

$$\psi'(x') = S(\Lambda)\psi(x),$$

where Λ is the Lorentz transformation matrix and $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$.

1.1 Show that covariance of the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

under Lorentz transformations implies that

$$S(\Lambda)\gamma^\nu S^{-1}(\Lambda) = \gamma^\mu \Lambda_\mu{}^\nu .$$

1.2 Show that $\bar{\psi}(x)$ transforms under Lorentz transformations as

$$\bar{\psi}'(x') = \bar{\psi}(x)\gamma^0 S^\dagger(\Lambda)\gamma^0 .$$

1.3 Argue that

$$S^{-1} = \gamma^0 S^\dagger \gamma^0 .$$

1.4 Show that $\bar{\psi}(x)\psi(x)$ is invariant under Lorentz transformations.

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PROBLEM 2

The Dirac equation for an electron in an electromagnetic field is given by

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi(x) = 0\,.$$

Assume that A_μ corresponds to a radial Coulomb potential, i.e., only $A_0 \neq 0$, and A_0 depends only on $\vec{x}^2$. The orbital angular momentum operator is $\vec{L} = \vec{x} \times \vec{p}$. The operators $\vec{p} = (p^1, p^2, p^3)$ are represented by $p^k = -i\partial/\partial x^k$. The spin angular momentum is given by $\vec{S} = \frac{1}{2}\gamma^5\gamma^0\vec{\gamma}$. The Hamiltonian H of the Dirac equation is

$$H = \gamma^0\,\vec{\gamma}\cdot\vec{p} + eA_0(x^2) + m\gamma^0\,.$$

$$\frac{\partial}{\partial x} f(x^2) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$$

$u = x^2$

2.1 Show that $[\vec{L},\ A_0(x^2)] = 0$.

2.2 Evaluate $[\vec{L},\ \vec{\gamma}\cdot\vec{p}]$.

2.3 Show that $[\vec{L},\ H] = i\gamma^0(\vec{\gamma}\times\vec{p})$.

2.4 Show that $[\vec{S}, \gamma^0] = 0$.

2.5 Show that $[\vec{L}+\vec{S},\ H] = 0$.

PROBLEM 3

The field $\phi(x)$, a solution of the Klein-Gordon equation, satisfies equal-time commutation relations

$$[\phi(x), \dot{\phi}(y)]_{x^0=y^0} = i\delta^3(\vec{x}-\vec{y})\,, \quad [\phi(x), \phi(y)]_{x^0=y^0} = 0\,.$$

where $\dot{\phi}(y) \equiv \partial\phi(y)/\partial y^0$.

3.1 What is the definition of the time-ordered product $T(\phi(x)\,\phi(y))$?

3.2 Let

$$G(x,y) = T(\phi(x)\,\phi(y))\,.$$

Show that

$$(\partial_x^2 + m^2)\,G(x,y) = -i\delta^4(x-y)$$

if $\phi(x)$ satisfies the Klein-Gordon equation.

3.3 A function $\phi_0(x)$ satisfies the equation

$$(\partial^\mu\partial_\mu + m^2)\,\phi_0(x) = j(x)\,,$$

for some function $j(x)$. Express $\phi_0(x)$ in terms of G and j.

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PROBLEM 4

Consider the scattering of a photon and a proton. The photon has energy E_1 and momentum $\vec{p}_1$, the proton of mass M is at rest. The spatial momentum of the photon is in the direction x^1. We will discuss only properties of the initial state.

4.1 What are the components of the four-momenta of the photon and the proton?

4.2 We perform a Lorentz transformation to the center-of mass frame, so that in the new system the total spatial momentum vanishes. In the new system the energy and momentum of the photon are E_3, p_3, of the proton E_4, p_4. Express E_3 and E_4 in terms of p_3 and M.

4.3 The Lorentz transformation required to do this is of the form

$$\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma\,,\ \Lambda^0{}_1 = \Lambda^1{}_0 = -\frac{\gamma v}{c}\,,\ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,.$$

Perform this Lorentz transformation on the initial four-momenta to express E_3, p_3, E_4, p_4 in terms of v/c, E_1 and M.

4.4 Express v/c in terms of E_1 and M with the relations obtained in parts (4.2) and (4.3).

$$\Lambda = \begin{bmatrix} \gamma & -\frac{\gamma v}{c} & 0 \\ -\frac{\gamma v}{c} & \gamma & \\ 0 & & 1_2 \end{bmatrix}$$